ON ORLICZ SEQUENCE SPACES

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ABSTRACT

It is proved that every Orlicz sequence space contains a subspace isomorphic to some l_p. The question of uniqueness of symmetric bases in Orlicz sequence spaces is investigated.

1. Introduction

The purpose of this paper is to investigate some linear topological properties of the Orlicz sequence spaces l_M defined with the aid of certain convex functions M (for precise definitions, see the next section). For $M(x) = x^p$; $1 \leqq p < +\infty$, the spaces l_M coincide with the classical sequence spaces l_p.

Our first result, Theorem 1 (Section 3), gives a new fragment of evidence to the general conjecture that every infinite dimensional Banach space contains a closed subspace isomorphic to c_0 or some l_p (see [4], [6] for a discussion of this and related conjectures). Theorem 1, though admittedly quite special, seems to be the first instance where the conjecture is verified for a class of spaces which a priori is not related to some given l_p-space. The "p" whose existence is shown in Theorem 1 is found by using the Schauder-Tychonoff fixed point theorem.

The second subject treated in the paper is the question of uniqueness of symmetric bases of a Banach space. Before explaining this problem, let us introduce some standard notions.

A sequence $\{x_n\}$ of a Banach space X is called a basis of X if every $x \varepsilon X$ has a unique expansion of the form $x = \sum_{n=1}^{\infty} \lambda_n x_n$. The basis is called normalized if $\| x_n \| = 1$ for all n. Two bases, $\{x_n\}$ of X and $\{y_n\}$ of Y are called equivalent if $\sum_{n=1}^{\infty} \lambda_n x_n$ converges if and only if $\sum_{n=1}^{\infty} \lambda_n y_n$ converges. Equivalently, this means the existence of an isomorphism T from X onto Y such that $Tx_n = y_n$; $n = 1, 2, \cdots$ A basis $\{x_n\}$ is called unconditional if for every permutation π of the integers the

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sequence $\{x_{\pi(n)}\}$ is also a basis. If, in addition, $\{x_{\pi(n)}\}$ is equivalent to $\{x_n\}$ for any such permutation then $\{x_n\}$ is called symmetric.

In the separable Orlicz sequence spaces l_M the natural unit vectors form a symmetric basis.

The general question of uniqueness of normalized bases originates in the desire to have a canonical representation of a Banach space as a sequence space in terms of a basis, which does not depend on the special choice of the basis. It has been proved that l_1, l_2 and c_0 are the only spaces with a unique, up to equivalence, normalized unconditional basis (c.f. [5] and its references). Furthermore, among all the spaces having an unconditional basis, these three spaces are the only ones in which every normalized unconditional basis is symmetric. Since normalized unconditional bases are so rarely unique it is natural to ask whether symmetric bases are unique up to equivalence. It is easily seen and well-known that the unit vector basis in c_0 or l_p is up to equivalence the only symmetric basis of these spaces. The usual proof of this fact (c.f. [8]) uses the "perfect homogeneity" of the natural unit vector basis of c_0 or l_p, a property which is characteristic only to these spaces [9].

In Section 4 we show that besides c_0 and l_p there are other Orlicz sequence spaces with a unique (up to equivalence) symmetric basis. On the other hand we construct an example of a reflexive Orlicz sequence space which fails to have this property. Thus, in general, a representation of a Banach space as a symmetric sequence space is not "canonical".

The study started in this section can be certainly continued in the direction of finding necessary and sufficient conditions on M which ensure the uniqueness of the symmetric basis of l_M. However, a more interesting problem raised by our examples is the question of finding out whether those spaces which have up to equivalence unique symmetric bases can be singled out among all Banach spaces with symmetric bases by any other important linear topological properties.

2. Preliminaries

In this section we summarize some material concerning Orlicz sequence spaces. This presentation follows essentially the Thesis of K. J. Lindberg [3] which contains a very clear and detailed exposition of most of the known facts regarding Orlicz sequence spaces as well as new results in the direction considered in this paper. (A standard reference for general Orlicz spaces is [2]).

An Orlicz function $M(x)$ is a continuous convex non-decreasing function defined for $x \geqq 0$ such that $M(0) = 0$ and $M(x) > 0$ for $x > 0$. To any Orlicz function $M(x)$ one can associate the space l_M of all sequences of scalars $\{x_n\}$, such that $\Sigma_{n=1}^{\infty} M(|x_n|/t) < \infty$ for some $t > 0$. The space l_M with the norm

$$\| \{x_n\} \| = \inf \left\{ t > 0;\ \sum_{n=1}^{\infty} M(|x_n|/t) \leqq 1 \right\}$$

becomes a Banach space which is called an Orlicz sequence space. We say that an Orlicz function $M(x)$ satisfies the Δ_2 condition (for small x) if for every $\lambda > 1$ there exists a constant $K(\lambda)$ and a positive number $x(\lambda)$ such that $M(\lambda x) \leqq K(\lambda) M(x)$ for $0 \leqq x \leqq x(\lambda)$. The words "for small x" will be omitted in the sequel.

It is easy to check that M satisfies the Δ_2 condition if and only if $\limsup_{x \to 0} xM'(x)/M(x) < +\infty$. (Recall that $M'(x)$ is a non-decreasing function defined for each $x \geqq 0$ except for a countable set of points in which we take $M'(x)$ as the derivative from the right). Other conditions each equivalent to the Δ_2 condition are (c.f. [1], [3]):

a) l_M does not contain a subspace isomorphic to l_∞.

b) The unit vectors form a symmetric basis of l_M.

c) l_M consists of all the sequences $\{x_n\}$ such that $\Sigma M(|x_n|) < \infty$.

For any Orlicz function M set

$$a_M = \liminf_{x \to 0} xM'(x)/M(x); \quad b_M = \limsup_{x \to 0} xM'(x)/M(x).$$

Then l_M is reflexive if and only if $a_M > 1$ and $b_M < +\infty$.

Two Orlicz functions $M_1(x)$ and $M_2(x)$ are called equivalent if there exist positive constants K_1, K_2, λ, μ and x_0 such that for all $0 \leqq x \leqq x_0$

$$K_1 \cdot M_2(\lambda x) \leqq M_1(x) \leqq K_2 \cdot M_2(\mu x)$$

When M_1 and M_2 satisfy the Δ_2-condition, they are equivalent if and only if the ratio $M_1(x)/M_2(x)$ is in the same time bounded and bounded away from zero on $(0, x_0)$ for some $x_0 > 0$. This is the case iff l_{M_1} and l_{M_2} consist of the same sequences i.e., the unit vector bases in l_{M_1} and l_{M_2} are equivalent.

We conclude this section by quoting the following result due to J. J. Lindberg [3].

Proposition 1. *Let X be an infinite-dimensional subspace of a separable Orlicz sequence space l_M. Then X has a subspace Y isomorphic to an Orlicz sequence space l_N for some Orlicz function N satisfying the Δ_2 condition. If X*

has an unconditional basis, Y may be chosen to be complemented in X, and if X has a symmetric basis then X itself is isomorphic to l_N.

3. In this section we prove the following result.

THEOREM 1. *Every Orlicz sequence space* l_M *contains a subspace isomorphic to* l_p *for some* $p \geqq 1$.

The theorem is trivial if M does not satisfy the Δ_2 condition since in this case l_M contains a subspace isomorphic to l_∞ and, therefore, it contains as a subspace any separable space, in particular, any l_p.

We can, therefore, assume that M satisfies the Δ_2 condition. First we need the following lemma.

LEMMA 1. *Let M be an Orlicz function satisfying the* Δ_2 *condition and let* C_M *be the subset of the Banach space* $C(0,1)$ *defined by* $C_M = \bigcap_{0<t\leqq 1} C_{M,t}$ *where* $C_{M\,t} = \overline{\mathrm{conv}}_{0<s<t}\{M(sx)/M(s)\}$ *(the closure being taken in the norm topology of* $C(0,1)$*).*

Then C_M *is a compact convex set containing the function* x^p *for some* $p \geqq 1$.

PROOF. As we have mentioned in the previous section, the Δ_2 condition insures the boundedness of the function $tM'(t)/M(t)$ in a certain neighborhood of zero and thus in the whole interval $(0,1)$. If $A = \sup_{0<t\leqq 1} tM'(t)/M(t)$ then for each $0 < s \leqq 1$

$$d/dx[M(sx)/M(s)] = sM'(sx)/M(s) \leqq sM'(s)/M(s) \leqq A, \qquad 0 < x \leqq 1,$$

and thus all the functions in $C_{M,t}$ are equicontinuous for any $t > 0$. In addition, they are uniformly bounded by 1; hence $C_{M,t}$ is compact for any $0 < t \leqq 1$. Consequently, C_M is a nonempty norm compact set consisting of Orlicz functions N satisfying $N(1) = 1$ and the Δ_2 condition with the same constant as for M.

Let $0 < r < 1$ and consider the map T_r defined on $C_{M,t}$: $0 < t < 1$ as follows:

$$T_r N(x) = N(rx)/N(r)$$

Since $N(r) \neq 0$ T_r is well defined. Moreover, if $N(x) = \sum_{i=1}^n \alpha_i\, M(s_i x)/M(s_i)$ with $\sum_{i=1}^n \alpha_i = 1$ and $0 < s_i \leqq t$ then

$$T_r N(x) = \sum_{i=1}^n \alpha_i \frac{M(s_i r x)}{M(s_i)} \Bigg/ \sum_{i=1}^n \alpha_i \frac{M(s_i r)}{M(s_i)}$$

$$= \sum_{i=1}^n \alpha_i \frac{M(s_i r)}{M(s_i)} \cdot \frac{M(s_i r x)}{M(s_i r)} \Bigg/ \sum_{i=1}^n \alpha_i \frac{M(s_i r)}{M(s_i)} \in C_{M,t}$$

It follows immediately that T_r maps C_M into C_M. The continuity of T_r on $C_{M,1}$ is checked easily by using the fact that all the functions from $C_{M,1}$ satisfy the Δ_2 condition with the *same* constant.

Using the Schauder-Tychonoff fixed point theorem for T_r on C_M we obtain the existence of a function $f \in C_M$ for which

$$(T_r f)(x) = f(xr)/f(r) = f(x); \qquad 0 \leqq x \leqq 1$$

By setting $q = \log f(r)/\log r$ we get that $f(r) = r^q$ and further $f(r^n) = (r^n)^q$; $n = 1, 2, \cdots$, since $f(r^n) = f^n(r)$. In addition, $q \geqq 1$ for f satisfies $f(r) \leqq r < 1$ as every other function from C_M does. Moreover, as we have shown in the beginning of this proof

$$|f(x) - f(y)| \leqq A|x - y|; \qquad x, y \in [0,1].$$

Thus, for every $x \in [0,1]$ we can choose an n such that $r^n < x \leqq r^{n-1}$ and have:

$$\begin{aligned} |f(x) - x^q| &\leqq |f(x) - f(r^n)| + |(r^n)^q - x^q| \\ &\leqq A|x - r^n| + (r^{n-1})^q - (r^n)^q \leqq 2A(1 - r) \end{aligned}$$

Now, if $r_k \to 1$, f_k a fixed point of T_{r_k} and $p_k = \log f_k(r_k)/\log r_k$ then

$$|f_k(x) - x^{p_k}| \leqq 2A(1 - r_k); \qquad 0 \leqq x \leqq 1$$

where $f_k \in C_M$; $k = 1, 2, \cdots$. Using a compactness argument for $\{f_k\}$ and $\{p_k\}$ and the fact that C_M is norm closed we obtain the existence of a $p \geqq 1$ such that $x^p \in C_M$. This concludes the proof of the lemma.

Proof of Theorem 1. We choose a number u, $0 < u < 1$ which will be fixed throughout this proof. According to Lemma 1 the compact convex set C_M contains a function x^p for some $p \geqq 1$ (where M is assumed to satisfy the Δ_2 condition). Thus $x^p \in C_{M,u}$ so we can find a convex combination

$$\sum_{i=1}^{n_1} \mu_i M(t_i x)/M(t_i); \qquad \sum_{i=1}^{n_1} \mu_i = 1;\ 0 < t_i < u;$$

$i = 1, 2, \cdots, n_1$ such that

$$\left| x^p - \sum_{i=1}^{n_1} \mu_i M(t_i x)/M(t_i) \right| < \tfrac{1}{2}; \qquad 0 \leqq x \leqq 1$$

Next, let j_1 be an integer for which $u^{j_1} < t_i$; $i = 1, 2, \cdots, n_1$. Since $x^p \in C_{M,u^{j_1}}$ we can find another convex combination $\sum_{i=n_1+1}^{n_2} \mu_i M(t_i x)/M(t_i)$; $\sum_{i=n_1+1}^{n_2} \mu_i = 1$; $0 < t_i \leqq u^{j_1}$; $i = n_1 + 1, \cdots, n_2$, such that

$$\left| x^p - \sum_{i=n_1+1}^{n_2} \mu_i M(t_i x)/M(t_i) \right| < \tfrac{1}{4}; \qquad 0 \leqq x \leqq 1.$$

Continuing in this manner we construct a decreasing sequence $\{t_i\}$, $\lim_{i\to\infty} t_i = 0$ and an increasing sequence of integers $\{j_k\}$; $\lim_{k\to\infty} j_k = +\infty$ such that $t_{n_k+1} \leqq u^{j_k} < t_{n_k}$; $k = 1,2,\cdots$, and

$$\left| x^p - \sum_{i=n_k+1}^{n_{k+1}} \mu_i M(t_i x)/M(t_i) \right| < \frac{1}{2^{k+1}}; \qquad 0 \leqq x \leqq 1$$

where

$$\sum_{i=n_k+1}^{n_{k+1}} \mu_i = 1; \qquad k = 1,2,\cdots.$$

Denote

$$\sigma_j = \{i;\, u^{j+1} \leqq t_i < u^j\}; \qquad \lambda_j = \sum_{i\in\sigma_j} \mu_i/M(t_i)\,.$$

Then

$$\sum_{i=n_k+1}^{n_{k+1}} \mu_i \frac{M(t_i x)}{M(t_i)} = \sum_{j=j_k}^{j_{k+1}-1} \sum_{i\in\sigma_j} \mu_i \frac{M(t_i x)}{M(t_i)}.$$

We have that

$$\sum_{j=j_k}^{j_{k+1}-1} \sum_{i\in\sigma_j} \frac{\mu_i}{M(t_i)} M(u^{j+1}x) \leqq \sum_{i=n_k+1}^{n_{k+1}} \mu_i \frac{M(t_i x)}{M(t_i)} \leqq \sum_{j=j_k}^{j_{k+1}-1} \sum_{i\in\sigma_j} \frac{\mu_i}{M(t_i)} M(u^j x)$$

or

$$\sum_{j=j_k}^{j_{k+1}-1} \lambda_j M(u^{j+1}x) \leqq \sum_{i=n_k+1}^{n_{k+1}} \mu_i \frac{M(t_i x)}{M(t_i)} \leqq \sum_{j=j_k}^{j_{k+1}-1} \lambda_j M(u^j x);$$

$$0 \leqq x \leqq 1.$$

Recall that the Δ_2 condition for M insures the boundedness of the expression $M(x/u)/M(x)$ in a neighborhood of zero and thus in the whole interval $[0,1]$ so that

$$\sup_{0<x\leqq 1} M(x/u)/M(x) = B < +\infty.$$

Consequently, for $0 \leqq x \leqq 1$,

$$\sum_{j=j_k}^{j_{k+1}-1} \lambda_j M(u^{j+1}x) \leqq \sum_{i=n_k+1}^{n_{k+1}} \mu_i \frac{M(t_i x)}{M(t_i)} \leqq B \sum_{j=j_k}^{j_{k+1}-1} \lambda_j M(u^{j+1}x);$$

and further

$$\sum_{j=j_k}^{j_{k+1}-1} \lambda_j M(u^{j+1}x) - \frac{1}{2^{k+1}} \leqq x^p \leqq \frac{1}{2^{k+1}} + B \sum_{j=j_k}^{j_{k+1}-1} \lambda_j M(u^{j+1}x).$$

If $[\lambda]$ denotes the largest integer $\leqq \lambda$, then

$$\sum_{j=j_k}^{j_{k+1}-1} [\lambda_j] M(u^{j+1}x) - \frac{1}{2^{k+1}} \leqq x^p \leqq$$

$$\leqq \frac{1}{2^{k+1}} + B \sum_{j=j_k}^{j_{k+1}-1} [\lambda_j] M(u^{j+1}x) + BM(1) \sum_{j=j_k}^{j_{k+1}-1} u^{j+1}.$$

which finally implies that for any sequence $\{a_k\}$ the series

$$\sum_{k=1}^{\infty} \sum_{j=j_k}^{j_{k+1}-1} [\lambda_j] \mathrm{M}(u^{j+1}|a_k|)$$

converges if and only if $\{a_k\} \in l_p$.

Let now $\{A_j\}_{j=1}^{\infty}$ be disjoint subsets of the integers so that A_j has exactly $[\lambda_j]$ elements ($A_j = \varnothing$ if $[\lambda_j] = 0$). Put

$$f_k = \sum_{j=j_k}^{j_{k+1}-1} u^{j+1} \left(\sum_{m \in A_j} e_m \right) \qquad k = 1, 2, \cdots$$

where e_m denotes the mth unit vector in l_M. The statement we just proved means that $\Sigma_k a_k f_k$ converges iff $\Sigma_k |a_k|^p < \infty$. In other words the closed linear span of $\{f_k\}_{k=1}^{\infty}$ is isomorphic to l_p. Q.E.D

COROLLARY. *Every closed subspace of a separable Orlicz sequence space l_M contains a subspace isomorphic to l_p for some $p \geqq 1$.*

PROOF. This follows from Proposition 1, Theorem 1 and the fact that separability of l_M implies that the Δ_2 condition holds for M.

REMARKS. Some particular cases of Theorem 1 have been proved by K. J. Lindberg in [3]. He has shown that if $a_M = b_M$ (for the definition, see Section 2 above) or if the function $xM'(x)/M(x)$ has a special asymptotic behavior then l_M contains a *complemented* subspace isomorphic to l_p for some $p \geqq 1$.

The analogue of Theorem 1 for Orlicz function spaces $L_M(0,1)$ is easy and well known. If M satisfies the Δ_2 condition (at ∞) then the Rademacher functions span a subspace of $L_M(0,1)$ which is isomorphic to l_2. If M does not satisfy Δ_2 at ∞ then $L_M(0,1)$ contains any separable space as a subspace.

4. Uniqueness of symmetric bases of Orlicz sequence spaces

We start this section by describing a class of Orlicz sequence spaces, including properly the l_p-spaces, which have (up to equivalence), a unique symmetric basis. In the statement of our result we refer again to the compact convex set C_M introduced in the previous section.

THEOREM 2. *Let l_M be a separable Orlicz sequence space for which the set C_M contains no Orlicz function equivalent to M. Then the usual unit vector basis $\{e_n\}$ is, up to equivalence, the unique symmetric basis of l_M.*

PROOF. Assume that f'_n is another symmetric basis of l_M. Thus, since $\{f'_n\}$ is equivalent to each of its subsequences, we can find by a standard procedure a normalized block-basis

$$f''_n = \sum_{i=q_n+1}^{q_n+1} t_i e_i, \quad \sum_{i=q_n+1}^{q_n+1} M(t_i) = \|f''_n\| = 1$$

which is equivalent to $\{f'_n\}$ (q_n is an increasing sequence tending to $+\infty$). Since the Δ_2 conditions holds for M it follows, as we have already mentioned in the proof of Lemma 1, that the functions $\sum_{i=q_n+1}^{q_n+1} M(t_i x)$; $n = 1,2,\cdots$, form a totaly bounded subset of $C(0,1)$. Hence we can assume without loss of generality that there exists an Orlicz function N, satisfying again the Δ_2 condition such that the sequence $\sum_{i=q_n+1}^{q_n+1} M(t_i x)$ tends to $N(x)$ uniformly on $[0,1]$.

Incidentally, this shows that $\{f'_n\}$ is equivalent to the unit vector basis $\{f_n\}$ of l_N.

Replacing M by N and $\{f'_n\}$ by $\{e_n\}$ and repeating our arguments we can construct a sequence of functions $\sum_{j=r_n+1}^{r_n+1} N(s_j x)$; $\sum_{j=r_n+1}^{r_n+1} N(s_j) = 1$; $n = 1,2,\cdots$, which converges uniformly on $[0,1]$ to an Orlicz function $M_1(x)$ and the basis $\{e_n\}$ is equivalent to the usual unit vector basis of l_{M_1} Therefore, it follows immediately that M_1 is equivalent to M.

Combining these facts one can easily show that $M_1(x)$ can be approximated uniformly on $[0,1]$ by sums having the form

$$\sum_{i=q_n+1}^{q_n+1} \sum_{j=r_{m_n}+1}^{r_{m_n+1}} M(t_i s_j x) = \sum_{i=q_n+1}^{q_n+1} \sum_{j=r_{m_n}+1}^{r_{m_n+1}} \frac{M(t_i s_j x)}{M(t_i s_j)} M(t_i s_j)$$

where the sums $\sum_{i=q_n+1}^{q_n+1} \sum_{j=r_{m_n}+1}^{r_{m_n+1}} M(t_i s_j)$ tend to 1.

Thus $M_1(x)$ can be approximated uniformly on $[0,1]$ by convex combinations of the functions $M(t_i s_j x)/M(t_i s_j)$. Consequently, if at least one of the sequences $\{t_i\}$ or $\{s_j\}$ tends to zero then $M_1 \in C_M$ which contradicts our hypothesis. Hence, we can assume the existence of a number $\alpha > 0$ and subsequences $\{t_{i_k}\}$ and $\{s_{j_k}\}$ for which $|t_{i_k}| \geqq \alpha$; $|s_{j_k}| \geqq \alpha$; $k = 1,2,\cdots$. In this case it follows easily that the bases $\{e_n\}$ and $\{f'_n\}$ are equivalent which completes the proof.

COROLLARY. *Let l_M be an Orlicz space for which $a_M = b_M$* (i.e. $\lim_{x\to 0} xM'(x)/$

$M(x)$ exists). Then the usual unit vector basis is, up to equivalence, the unique symmetric basis of l_M.

PROOF. If $p = a_M = b_M$; then for any $\varepsilon > 0$ there is $x_\varepsilon > 0$ such that

$$\frac{p-\varepsilon}{x} \leqq \frac{M'(x)}{M(x)} \leqq \frac{p+\varepsilon}{x}; \qquad 0 < x < x_\varepsilon$$

Integrating this inequality between tx and t for some $0 < t < x_\varepsilon$ and $0 < x < 1$ we get

$$x^{p+\varepsilon} \leqq \frac{M(xt)}{M(t)} \leqq x^{p-\varepsilon}$$

from which we can easily conclude that C_M contains only one function namely x^p. Thus, either M is not equivalent to x^p and then we conclude the proof by using Theorem 2, or M is equivalent to x^p in which case this result is known as we have pointed out is Section 1.

EXAMPLE. For each $p \geqq 1$, the function $M(x) = x^p/(1 - \log x)$ satisfies the conditions of the previous corollary without being equivalent to x^p. There are also examples of functions M such that $a_M \neq b_M$ but M satisfies the assumption of Theorem 2.

The remaining part of this section is devoted to the construction of an example of an Orlicz sequence space having at least two non-equivalent symmetric bases. We shall need the following Lemma:

LEMMA 2. *Let $H(x)$ and $K(x)$ be two continuous non-decreasing convex functions defined on an interval $[t,1]$ for some $0 < t < 1$ and such that:*

1° $H(1) = K(1) = 1;\ 1 > H(t) > 0;\ 1 > K(t) > 0$

2° *There exist numbers $d > c > 1$ such that*

$H'(1) = K'(1) = c$ and

$$c \leqq xH'(x)/H(x) \leqq d;\ c \leqq xK'(x)/K(x) \leqq d$$

for every $x \in [t,1]$ in which both derivatives exist (othcrwise we take instead the right derivates). Then the function

$$\tilde{H}(x) = \begin{cases} H(x) & t \leqq x \leqq 1 \\ H(t)K(x/t); & t^2 \leqq x < t \end{cases}$$

is continuous, convex, non-decreasing and satisfies 2° on $[t^2, 1]$ with the same c and d.

PROOF. It is obvious that $\tilde{H}$ is a continuous non-decreasing extension of H. For $t^2 \leqq x < t$ we have

$$\tilde{H}'(x) = H(t)K'(x/t)/t,$$

and in particular

$$\tilde{H}'(t-0) = H(t)K'(1)/t \leqq H'(t),$$

which together imply the convexity of $\tilde{H}$ on $[t^2, 1]$. Finally,

$$\frac{x\tilde{H}'(x)}{\tilde{H}(x)} = \frac{(x/t)K'(x/t)}{K(x/t)}; \qquad t^2 \leqq x < t,$$

shows that the inequalities from 2° continue to hold.

THEOREM 3. *There exists a reflexive Orlicz space l_M having at least two non-equivalent symmetric bases.*

PROOF. We begin by constructing two functions, $M(x)$ and $N(x)$ defined on $[\frac{1}{2}, 1]$ satisfying all the requirements of the preceding Lemma and also that $M(\frac{1}{2})/N(\frac{1}{2}) = \lambda < 1$. The possibility of finding such a pair of functions is evident

By an inductive procedure we extend M and N to Orlicz functions on $[0,1]$. Put $t_n = 2^{-2^{n-1}}$, $n = 1, 2, \cdots$. Assume we have already extended M and N (the extensions will be denoted always by the same letter) to $[t_{3n+1}, 1]$ so that all the conditions in Lemma 2 are satisfied on this interval and in addition

$$(*) \qquad M(t_{3n+1})/N(t_{3n+1}) = \lambda^{2^{n-1}}$$

We extend now in three steps the functions to the interval $[t_{3n+4}, 1]$ as follows:

$$\left.\begin{aligned} M(x) &= M(t_{3n+1})N(x/t_{3n+1}) \\ N(x) &= N(t_{3n+1})N(x/t_{3n+1}) \end{aligned}\right\} t_{3n+1} \geqq x \geqq t_{3n+2} = t_{3n+1}^2,$$

$$\left.\begin{aligned} M(x) &= M(t_{3n+2})M(x/t_{3n+2}) \\ N(x) &= N(t_{3n+2})N(x/t_{3n+2}) \end{aligned}\right\} t_{3n+2} \geqq x \geqq t_{3n+3} = t_{3n+2}^2,$$

$$\left.\begin{aligned} M(x) &= M(t_{3n+3})M(x/t_{3n+3}) \\ N(x) &= N(t_{3n+3})M(x/t_{3n+3}) \end{aligned}\right\} t_{3n+3} \geqq x \geqq t_{3n+4} = t_{3n+3}^2.$$

It follows from Lemma 2 that M and N continue to satisfy all the requirements appearing in its statement on the interval $[t_{3(n+1)+1}, 1]$. Also (*) is satisfied for $n+1$ since

$$\frac{M(t_{3n+4})}{N(t_{3n+4})} = \frac{M(t_{3n+3})}{N(t_{3n+3})} = \left[\frac{M(t_{3n+2})}{N(t_{3n+2})}\right]^2 = \left[\frac{M(t_{3n+1})}{N(t_{3n+1})}\right]^2 = \lambda^{2^n}.$$

Thus by setting $M(0) = N(0) = 0$ we get Orlicz functions on $[0,1]$ which satisfy the Δ_2 condition and for which $\lim_{n\to\infty} M(t_n)/N(t_n) = 0$, i.e. $M(x)$ and $N(x)$ are not equivalent.

By the inductive construction we have

$$N(x) = M(t_{3n+1}x)/M(t_{3n+1}), \ t_{3n+1} \leqq x \leqq 1$$

and

$$M(x) = N(t_{3n+3}x)/N(t_{3n+3}), \ t_{3n+3} \leqq x \leqq 1.$$

By using the fact that N and M satisfy Δ_2 it follows that there are subsequences $\{s_m\}$ and $\{r_m\}$, of $\{t_{3n+1}\}$ and $\{t_{3n+3}\}$ respectively, so that for every $0 \leqq x \leqq 1$

$$\left| N(x) - M(s_m x)/M(s_m) \right| \leqq 2^{-2m}, \ \left| M(x) - N(r_m x)/N(r_m) \right| \leqq 2^{-2m}.$$

Since $\lim_{m\to\infty} M(s_m) = \lim_{m\to\infty} N(r_m) = 0$ while M and N are bounded above by 1 we may assume (by passing again to a subsequence, if necessary) that

$$\left| N(x) - \lambda_m M(s_m x) \right| \leqq 2^{-m}, \quad \left| M(x) - \gamma_m N(r_m x) \right| \leqq 2^{-m}$$

for all $0 \leqq x \leqq 1$, where $\lambda_m = [1/M(s_m)]$ and $\gamma_m = [1/N(r_m)]$.

It follows that the series $\Sigma_m N(|a_m|)$ converges iff $\Sigma_m \lambda_m M(s_m |a_m|)$ converges while $\Sigma_m M(|b_m|)$ converges iff $\Sigma_m \gamma_m N(r_m |b_m|)$ converges. Let $\{A_m\}$ be disjoint subsets of the integers having exactly λ_m elements, and let $\{e_n\}$ denote the unit vectors in l_M. The subspace U of l_M spanned by the vectors $y_m = \lambda_m \Sigma_{n\in A_m} e_n$ is thus isomorphic to l_N. Since U is complemented in l_M (cf. [5]) it follows that l_N is isomorphic to a complemented subspace of l_M i.e. $l_M \approx l_N \oplus X$ for some space X. Similarly $l_N \approx l_M \oplus Y$ for some Banach space Y. An application of Pełczyński's decomposition method [7] shows that $l_N \approx l_M$. Indeed,

$$l_N \approx l_M \oplus Y \approx l_M \oplus l_M \oplus Y \approx l_M \oplus l_N \approx l_N \oplus l_N \oplus X \approx l_N \oplus X \approx l_M.$$

This completes the proof of the theorem.

Remarks The construction used here allows a great degree of freedom in choosing N and M. For example, given any $1 < a < b < \infty$ we can construct an M (and N) as above so that $a_M = a$, $b_M = b$ and, of course, l_M does not have a unique symmetric basis. This should be compared to the Corollary to Theorem 2.

An easy modification of the construction above shows that there is an Orlicz space which has at least three symmetric mutually nonequivalent bases. One simply has to construct inductively in an obvious manner three functions $M_1(X)$, $M_2(X)$, $M_3(X)$ instead of the two functions $M(X)$ and $N(X)$. In fact, the same procedure enables the construction of an Orlicz sequence space l_M which has a countably infinite number of mutually non-equivalent symmetric bases. We do not know whether there exists an Orlicz sequence space which has exactly two (or any other finite number) of mutually non-equivalent symmetric bases. It may be that the existence of two non-equivalent symmetric bases implies already the existence of infinitely many non-equivalent ones.

REFERENCES

1. Y. Gribanov, *On the theory of l_M spaces*, Ucen. Zap. Kazansk. un-ta **117**, (1957), 62–65 (Russian).

2. M. A. Krasnoselskii and Ya. B. Rutickii, *Convex Functions and Orlicz Spaces*, Groningen (The Netherlands), 1961 (translated from Russian).

3. K. J. Lindberg, *Contractive projections in Orlicz sequence spaces and continuous function spaces*, Ph.D. thesis, Univetsity of California, Berkeley, 1970.

4. J. Lindenstrauss, *Some aspects of the theory of Banach spaces*, Advances in Math. **5** (1970) 159–180.

5. J. Lindenstrauss and M. Zippin, *Banach spaces with a unique unconditional basis*, J. Functional Analysis **3** (1969), 115–125.

6. V. D. Milman, *Geometric theory of Banach spaces I*, Russian Math. Surveys **25** (1970) 111–170.

7. A. Pełczyński, *On the isomorphism of the spaces m and M*, Bull. Acad. Polon. Sci. **6** (1958), 695–696.

8. I. Singer, *Bases in Banach Spaces* I, Springer Verlag, 1970.

9. M. Zippin, *On perfectly homogeneous bases in Banach spaces*, Israel J. Math. **4** (1966), 265–272.

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